Logarithmic Functions

Fix $a > 0$, $a \neq 1$.

Definition The logarithmic function with base a, denoted $\log_a$, is defined by

$$y = \log_a(x) \iff x = a^y$$

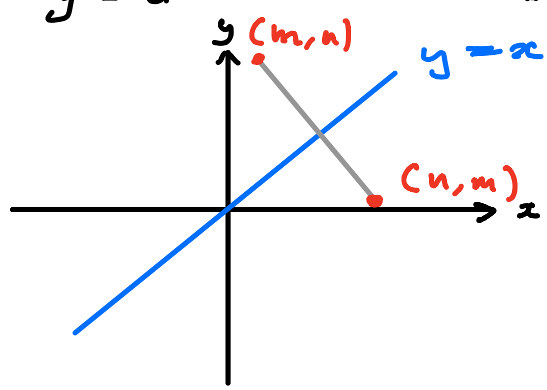
if and only if.

Example : $8 = 2^3 \Rightarrow \log_2(8) = 3$

Basic Properties : $a^{\log_a(x)} = x$, $\log_a(a^x) = x$, $\log_a(1) = 0$.

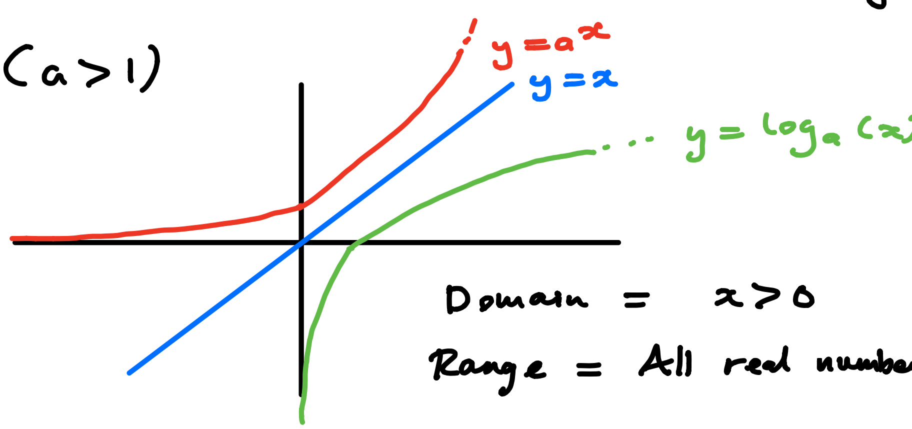
Graph of $y = \log_a(x)$ = Graph of $x = a^y$

= Graph of $y = a^x$ with x and y switched



$\Rightarrow$ Graph of $y = \log_a(x)$ = Graph of $y = a^x$ reflected in line $y = x$

Picture ($a > 1$)



Domain = $x > 0$

Range = All real numbers

Most important logarithmic function : $\ln = \log_e$

"LN" base e

Laws of Exponentials $\Rightarrow$ Laws of Logarithms $(a \neq 1)$

1. $\log_a(bc) = \log_a(b) + \log_a(c)$
2. $\log_a(b^c) = c\log_a(b)$

Warning : There is no law for $\log_a(b+c)$.

$\Rightarrow$ $\ln(bc) = \ln(b) + \ln(c)$ $\quad \forall\, b, c > 0$

$\ln(b^c) = c\ln(b)$

We will almost exclusively use $\ln$ in this course

Example : 1. Find a solution to $3^x = 7$.

$$3^x = 7 \Rightarrow \ln(3^x) = \ln(7)$$

$$\Rightarrow x\ln(3) = \ln(7) \Rightarrow x = \frac{\ln(7)}{\ln(3)}.$$

2. $y = \log_a(x) \Leftrightarrow x = a^y$

$$\Leftrightarrow \ln(x) = \ln(a^y) \Leftrightarrow \ln(x) = y\ln(a)$$

$$\Leftrightarrow y = \frac{\ln(x)}{\ln(a)}$$

Hence $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Conclusion : We only really need $\ln$.

Example A savings account has annual interest rate of 5% (ie $r = 0.05$). If interest is compounded continuously determine how long it will take for the amount in account to triple.

P = initial deposit

$f(t)$ = amount in account at time t (in years)

$\Rightarrow \quad f(t) = P e^{0.05t}$

Need t such that $f(t) = 3P = 3f(0)$

$3P = P e^{0.05t} \Rightarrow 3 = e^{0.05t}$

$\Rightarrow \quad \ln(3) = \ln(e^{0.05t}) = 0.05t \ln(e)$

$\Rightarrow \quad t = \dfrac{\ln(3)}{0.05}$ years.